Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Solids and Structures 42 (2005) 3700–3715

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The effect of rotation on the nonlinear magnetoelastic response of a circular cylindrical tube

A. Dorfmann [a], R.W. Ogden [b,*], G. Saccomandi [c]

[a] Centre of Mechanics and Structural Dynamics, Vienna University of Technology, Wiedner Hauptstrasse 8-10/E2063, 1040 Vienna, Austria

[b] Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

[c] Sezione di Ingegneria Industriale, Dipartimento di Ingegneria dell'Innovazione, Università degli Studi di Lecce, Lecce 73100, Italy

Received 20 October 2004; received in revised form 24 November 2004
Available online 21 January 2005

Abstract

In this paper we use a recently developed concise general theory of nonlinear magnetoelasticity to analyze the mechanical response of (a) a circular cylindrical tube under steady rotation about its axis in an azimuthal magnetic field, and (b) a solid circular cylinder also under steady rotation about its axis in an axial magnetic field. It is found that for problem (a) the magnetic field can either enhance or counteract the effect of rotation, while for problem (b) the magnetic field reinforces the effect of rotation.

Keywords: Nonlinear magnetoelasticity; Magneto-sensitive elastomers; Rotational deformation; Magnetoelastic cylinder

1. Introduction

In a recent series of papers (Dorfmann and Ogden, 2003a,b; Dorfmann and Ogden, 2004, in press) we have developed several alternative formulations of the equations of nonlinear magnetoelasticity and applied those equations to the solution of a number of boundary-value problems. The equations have intrinsic mathematical interest but the work is primarily motivated by its application to the response of magneto-sensitive (MS) elastomers. In the present paper we make use of a particularly simple formulation described

* Corresponding author. Tel.: +44 141 330 4550; fax: +44 141 330 4111.
E-mail addresses: dorfmann@mail.boku.ac.at (A. Dorfmann), rwo@maths.gla.ac.uk (R.W. Ogden), giuseppe.saccomandi@unile.it (G. Saccomandi).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.11.019

by Dorfmann and Ogden (2004, in press) in the analysis of two problems of practical interest for an incompressible isotropic magnetoelastic solid. These are for (a) a thick-walled circular cylindrical tube, and (b) a solid circular cylinder, each undergoing a steady rotation about its axis. For (a) the rotation takes place in the presence of an azimuthal magnetic field, while for (b) the magnetic field is axial.

In Section 2 we summarize the mechanical and magnetic balance equations, while in Section 3 the constitutive law for a magnetoelastic solid is developed, both in general and for the particular case of an isotropic magnetoelastic material, and for both compressible and incompressible materials. We include formulations of the constitutive law based separately on the magnetic field and the magnetic induction as the independent magnetic variable.

In Section 4 problem (a) is analyzed. In particular, we obtain an explicit formula that relates the rotation, the magnetic field and the deformation through an integral involving the energy function. We point out that this is similar in structure to the problem of inflation of an elastic tube under an internal pressure. Because of the differences in the constitutive law as compared with a purely elastic material, however, the effect of the magnetic field can be different from that of a pressure. In particular, if the length of the tube is maintained under a rotation (without a magnetic field) by the application of an axial load then this load may be positive or negative, depending on the form of constitutive law. The presence of the magnetic field can either enhance or counteract the effect of rotation, depending on the magnetic contribution to the constitutive law. Under reasonable assumptions on the latter the axial load in the absence of rotation can be negative when an azimuthal magnetic field is applied, thereby preventing the tube from lengthening. Thus, the magnetic field alone, without an applied axial load, would induce a magnetostrictive increase in length.

The situation is somewhat different for problem (b), which is examined briefly in Section 5. In this case, because the magnetic field considered is axial, under the same constitutive assumptions as for (a), the magnetic field enhances the effect of rotation. Thus, without the rotation, the magnetic field generates a shortening of the cylinder. This is consistent with the discussion of the problem of extension of a cylinder in an axial magnetic field due to Kankanala and Triantafyllidis (2004), who considered a special form of constitutive law. Section 6 contains some closing remarks.

2. Basic equations

2.1. Kinematics

Consider a magneto-sensitive body occupying a region $\mathscr{B}_0$ in a three-dimensional Euclidean space in the absence of mechanical loads and magnetic fields. We assume that the body is stress free in this configuration. Let material points in $\mathscr{B}_0$ be labelled by their position vectors $\mathbf{X}$ relative to some chosen origin, and denote time by $t \in I \subset \mathbb{R}$, where I is an interval of $\mathbb{R}$.

Suppose that the material is now deformed by the application of mechanical loads and a magnetic field such that the point $\mathbf{X}$ moves to a new position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in the resulting configuration, which we denote by $\mathscr{B}_t$. The vector field $\boldsymbol{\chi}$ describes the motion of the body and is defined for $t \in I$ and for $\mathbf{X} \in \mathscr{B}_0 \cup \partial\mathscr{B}_0$, where the boundary of $\mathscr{B}_0$ is denoted by $\partial\mathscr{B}_0$.

The time-dependent deformation gradient tensor relative to $\mathscr{B}_0$, denoted $\mathbf{F}$, and its determinant, denoted J, are

$$\mathbf{F} = \text{Grad}\,\boldsymbol{\chi}, \quad J = \det \mathbf{F} > 0, \tag{1}$$

respectively, where Grad is the gradient operator with respect to $\mathbf{X}$. The Cartesian components of $\mathbf{F}$ are given by $F_{ij} = \partial x_i/\partial X_j$, where X_i and x_i, $i = 1, 2, 3$, are the Cartesian components of $\mathbf{X}$ and $\mathbf{x}$, respectively. For full details of the kinematics of solid continua we refer to, for example, Ogden (1997) and Holzapfel (2001).

The velocity, denoted $\mathbf{v}$, and acceleration, denoted $\mathbf{a}$, of a material point $\mathbf{X}$ are defined as

$$\mathbf{v} \equiv \mathbf{x}_{,t} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{a} \equiv \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{2}$$

where $\partial/\partial t$ is the material time derivative, denoted for brevity by $_{,t}$.

2.2. Magnetic balance equations

In the current configuration $\mathscr{B}_t$ the relevant magnetic field variables are $\mathbf{H}$, the magnetic field, $\mathbf{B}$, the magnetic induction, and $\mathbf{M}$, the magnetization. These are related by the standard equation

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{3}$$

where μ_0 is the magnetic permeability in vacuo. In vacuo, $\mathbf{M} = \mathbf{0}$ and Eq. (3) reduces to $\mathbf{B} = \mu_0\mathbf{H}$. For time-independent magnetic fields, to which we restrict attention here, the vectors $\mathbf{B}$ and $\mathbf{H}$ satisfy the appropriate specializations of Maxwell's equations, which are

$$\mathrm{div}\mathbf{B} = 0, \quad \mathrm{curl}\mathbf{H} = \mathbf{0}, \tag{4}$$

where, respectively, div and curl are the divergence and curl operators with respect to $\mathbf{x}$. A further connection between these magnetic field vectors will be introduced in the form of a constitutive law in Section 3. Then, if, for example, the magnetization $\mathbf{M}$ of the material is given by a constitutive law as a function of $\mathbf{B}$, Eq. (3) determines $\mathbf{H}$, also as a function of $\mathbf{B}$.

Eqs. (3) and (4) are expressed in Eulerian form. Lagrangian forms for the magnetic induction and the magnetic field, denoted by $\mathbf{B}_l$ and $\mathbf{H}_l$ respectively, are obtained by pull-back operations from $\mathscr{B}_t$ to $\mathscr{B}_0$. These Lagrangian fields are related to $\mathbf{B}$ and $\mathbf{H}$ through

$$\mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_l, \tag{5}$$

where $\mathbf{F}^{-\mathrm{T}} = (\mathbf{F}^{-1})^{\mathrm{T}}$ and T denotes the transpose of a second-order tensor. For derivations of these connections we refer to, for example, Dorfmann and Ogden (2003b, 2004) and Steigmann (2004).

The standard identities

$$\mathrm{Div}(J\mathbf{F}^{-1}\mathbf{B}) = J\,\mathrm{div}\,\mathbf{B}, \quad \mathrm{Curl}(\mathbf{F}^{\mathrm{T}}\mathbf{H}) = \mathrm{curl}\,\mathbf{H} \tag{6}$$

ensure that the Maxwell equations (4) are equivalent (for suitably regular deformations) to

$$\mathrm{Div}\,\mathbf{B}_l = 0, \quad \mathrm{Curl}\,\mathbf{H}_l = \mathbf{0}, \tag{7}$$

where, respectively, Div and Curl are the divergence and curl operators with respect to $\mathbf{X}$.

However, no corresponding simple transformation between $\mathbf{M}$ and $\mathbf{M}_l$ arises naturally in a similar way, but, in view of the fact that $\mathbf{H}$ and $\mathbf{M}$ occur as a sum in (3), a Lagrangian form of $\mathbf{M}$, denoted $\mathbf{M}_l$, may be defined, similarly to $(5)_2$, by

$$\mathbf{M}_l = \mathbf{F}^{\mathrm{T}}\mathbf{M}. \tag{8}$$

(Note, however, that this is not the only possible definition of a Lagrangian form of $\mathbf{M}$, since $\mathbf{M}_l$ could equally be defined similarly to $\mathbf{B}_l$.) On use of (5) and (7) in (3) we obtain

$$J^{-1}\mathbf{c}\mathbf{B}_l = \mu_0(\mathbf{H}_l + \mathbf{M}_l), \tag{9}$$

where $\mathbf{c}$ is the right Cauchy-Green deformation tensor defined by $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

2.3. Mechanical balance laws

Let ρ_0 and ρ denote the mass densities in the reference and deformed configurations, $\mathscr{B}_0$ and $\mathscr{B}_t$, respectively. Then, for the considered continuum, the conservation of mass may be written in the standard form

$$J\rho = \rho_0, \tag{10}$$

which we employ here.

In the absence of mechanical body forces, the equation of motion may be written in the simple form

$$\text{div}\boldsymbol{\tau} = \rho\mathbf{a}, \tag{11}$$

where $\boldsymbol{\tau}$ is the total stress tensor (see, for example, Dorfmann and Ogden (2003a,b, 2004) Steigmann (2004), Kovetz (2000), Hutter and van de Ven (1978) and Maugin (1988) for discussion of different stress tensors and magnetic body forces). Balance of angular momentum requires that $\boldsymbol{\tau}$ be symmetric:

$$\tau^{\mathrm{T}} = \tau. \tag{12}$$

If the governing equations are expressed in Eulerian form, as above, then the field variables $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$ and $\boldsymbol{\tau}$ are defined in the current configuration as functions of the position vector $\mathbf{x}$. These equations can also be recast in Lagrangian form. For this purpose we define a total nominal stress tensor $\mathbf{T}$ related to $\boldsymbol{\tau}$ by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{13}$$

The equation of motion (11) is then expressible equivalently as

$$\text{Div}\mathbf{T} = \rho_0\mathbf{a}. \tag{14}$$

2.4. Boundary conditions

To complete the formulation of boundary-value problems we need to supplement the governing differential equations with appropriate constitutive laws (which are considered in Section 3) and boundary conditions. The magnetic boundary conditions may be given in either Lagrangian or Eulerian form. Here, however, we confine attention to the latter. At a bounding surface of the considered material in the deformed configuration the vector fields $\mathbf{B}$ and $\mathbf{H}$ satisfy the standard jump conditions

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \quad \mathbf{n} \cdot [\mathbf{H}] = \mathbf{0}, \tag{15}$$

where the square brackets indicate a discontinuity across the surface and $\mathbf{n}$ is the unit normal to the surface, which, by convention, is taken as the outward pointing normal to the boundary. In the case of $(15)_2$ it is assumed that there are no surface currents.

At the boundary of the body the (total) traction continuity condition is (in Eulerian form)

$$[\boldsymbol{\tau}]\mathbf{n} = \mathbf{0}, \tag{16}$$

where, at the exterior of the body boundary, the traction includes the appropriate Maxwell stress (in the case of a vacuum, for example) and any applied mechanical tractions.

Boundary conditions in which the position is prescribed on part of the boundary may also be prescribed, but we do not make them explicit here.

3. Constitutive equations

To derive constitutive equations for the total stress tensor $\boldsymbol{\tau}$ and the magnetization $\mathbf{M}$, we assume the existence of a free energy function, denoted Ψ and defined per unit mass, with $\mathbf{F}$ and $\mathbf{B}$ as independent variables. Use is then made of the connection $(5)_1$ so that $\mathbf{F}$ and $\mathbf{B}_l$ are the independent variables and the free energy function denoted Φ. Thus, we write

$$\Psi = \Psi(\mathbf{F}, \mathbf{B}) \equiv \Psi(\mathbf{F}, J^{-1}\mathbf{F}\mathbf{B}_l) \equiv \Phi(\mathbf{F}, \mathbf{B}_l). \tag{17}$$

Objectivity requirements (see, for example, Steigmann (2004) and Dorfmann and Ogden (2003b)) enable Φ to be treated as a function of $\mathbf{c}$ (instead of $\mathbf{F}$) and $\mathbf{B}_l$.

3.1. Compressible materials

3.1.1. Use of $\mathbf{B}_l$ as an independent variable

For unconstrained materials, we have (see Dorfmann and Ogden (2004, in press))

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}} + \mu_0^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right], \tag{18}$$

where $\mathbf{I}$ is the identity tensor, and symmetry of $\boldsymbol{\tau}$ follows automatically from objectivity of Φ.

In terms of Φ the (Eulerian) magnetization vector is given by

$$\mathbf{M} = -\rho J\mathbf{F}^{-\mathrm{T}}\frac{\partial\Phi}{\partial\mathbf{B}_l}. \tag{19}$$

In vacuum, $\Phi \equiv 0$, and the stress $\boldsymbol{\tau}$ reduces to the Maxwell stress, here denoted $\boldsymbol{\tau}_m$, given by

$$\boldsymbol{\tau}_m = \mu_0^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right], \tag{20}$$

with $\mathbf{B} = \mu_0\mathbf{H}$. On use of (4) it then follows that $\operatorname{div}\boldsymbol{\tau}_m = \mathbf{0}$.

The corresponding Lagrangian forms of the total stress tensor and the magnetization vector are given by

$$\mathbf{T} = \rho_0\frac{\partial\Phi}{\partial\mathbf{F}} + \mu_0^{-1}\left[\mathbf{B}_l\otimes\mathbf{B} - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})J\mathbf{F}^{-1}\right] \tag{21}$$

and

$$\mathbf{M}_l = -\rho_0\frac{\partial\Phi}{\partial\mathbf{B}_l}. \tag{22}$$

On use of the amended free energy function introduced by Dorfmann and Ogden (2004), denoted $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$ and defined by

$$\Omega = \rho_0\Phi + \frac{1}{2}\mu_0^{-1}J^{-1}\mathbf{B}_l\cdot(\mathbf{c}\mathbf{B}_l), \tag{23}$$

we may write the total stress tensors $\boldsymbol{\tau}$ and $\mathbf{T}$ in the compact forms

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}. \tag{24}$$

Note that these equations apply only within the material since $\mathbf{F}$ is not defined outside.

On use of Eqs. (23) and (9), which is based on (3), together with (22) we obtain

$$\mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \tag{25}$$

and, for a given $\mathbf{B}_l$, the magnetization $\mathbf{M}_l$ is then given by (9) and (25). As for (24), Eq. (25) applies only within the material.

3.1.2. Use of $\mathbf{H}_l$ *as an independent variable*

As in Dorfmann and Ogden (2004, in press) we now consider an alternative formulation of the governing equations based on use of $\mathbf{H}$ (or, equivalently, $\mathbf{H}_l$) as the independent magnetic variable. For this purpose we make use of the Legendre transformation

$$\Omega^*(\mathbf{F}, \mathbf{H}_l) = \Omega(\mathbf{F}, \mathbf{B}_l) - \mathbf{H}_l \cdot \mathbf{B}_l, \tag{26}$$

which defines $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ as a function of $\mathbf{F}$ and $\mathbf{H}_l$ (under suitable invertibility assumptions on (25)). Then, the counterparts of Eqs. $(24)_2$ and (25) are

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}, \tag{27}$$

with $\mathbf{M}_l$ given by

$$\mathbf{M}_l = \mu_0^{-1} J^{-1} \mathbf{c} \mathbf{B}_l - \mathbf{H}_l. \tag{28}$$

Alternatively, instead of defining $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ via (26), and to avoid the need for using the Legendre transform, an energy defined as a function of $\mathbf{F}$ and $\mathbf{H}$ or $\mathbf{F}$ and $\mathbf{H}_l$ could be defined ab initio and the constitutive law then developed in terms of $\mathbf{H}_l$.

In either formulation the Eqs. (7), or equivalently (4), must be satisfied.

3.2. Incompressible materials

For an incompressible material we have the equivalent constraints

$$\det \mathbf{F} \equiv 1, \quad \rho \equiv \rho_0, \tag{29}$$

and the amended free energy function then simplifies to

$$\Omega = \rho_0 \Phi + \frac{1}{2} \mu_0^{-1} \mathbf{B}_l \cdot (\mathbf{c} \mathbf{B}_l). \tag{30}$$

The expressions for the total stress tensors (24) now include the Lagrange multiplier p associated with the constraint (29) and are given by

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{31}$$

in terms of Ω and

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{F}^{-1}, \tag{32}$$

in terms of Ω^*, where we have used p^* for the Lagrange multiplier since in general it will not be the same as p.

Eqs. (25) and $(27)_2$ are unchanged, except that in (33) the constraint (29) is in force. For convenience we collect these together here as

$$\mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}, \quad \mathbf{B}_l = -\frac{\partial \Omega^*}{\partial \mathbf{H}_l}. \tag{33}$$

3.3. Isotropy

We note that the symmetry considerations for magnetoelastic materials are similar to those arising for transversely isotropic elastic solids. Working in terms of the formulation based on Ω and following the analysis of transversely isotropic elastic solids in Spencer (1971) and Ogden (2001), for example, we define an *isotropic* magnetoelastic material as one for which Ω is an isotropic function of the two tensors $\mathbf{c}$ and $\mathbf{B}_l \otimes \mathbf{B}_l$. In this case the form of Ω is reduced to dependence on the principal invariants I_1, I_2, I_3 of $\mathbf{c}$, defined by

$$I_1 = \mathrm{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\mathbf{c})^2 - \mathrm{tr}(\mathbf{c}^2)\right], \quad I_3 = \det\mathbf{c} = J^2, \tag{34}$$

where tr is the trace of a second-order tensor, combined with the additional invariants involving $\mathbf{B}_l$, defined by

$$I_4 = |\mathbf{B}_l|^2, \quad I_5 = (\mathbf{c}\mathbf{B}_l)\cdot\mathbf{B}_l, \quad I_6 = (\mathbf{c}^2\mathbf{B}_l)\cdot\mathbf{B}_l. \tag{35}$$

The explicit form of $\boldsymbol{\tau}$ for an incompressible isotropic magnetoelastic material is

$$\boldsymbol{\tau} = 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5\mathbf{B}\otimes\mathbf{B} + 2\Omega_6(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}), \tag{36}$$

while the magnetic field is given by

$$\mathbf{H} = 2(\Omega_4\mathbf{b}^{-1}\mathbf{B} + \Omega_5\mathbf{B} + \Omega_6\mathbf{b}\mathbf{B}), \tag{37}$$

where $I_3 \equiv 1$, $\Omega = \Omega(I_1, I_2, I_4, I_5, I_6)$, the subscripts 1, 2, 4, 5, 6 signify partial differentiation with respect to I_1, I_2, I_4, I_5, I_6, respectively, and $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy-Green deformation tensor.

In the formulation based on Ω^*, we need to replace the invariants (35) by corresponding invariants using $\mathbf{H}_l$. For these, we choose to use the invariants K_4, K_5, K_6 defined by

$$K_4 = |\mathbf{H}_l|^2, \quad K_5 = (\mathbf{c}\mathbf{H}_l)\cdot\mathbf{H}_l, \quad K_6 = (\mathbf{c}^2\mathbf{H}_l)\cdot\mathbf{H}_l. \tag{38}$$

Then, the total stress is

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{b}\mathbf{H}\otimes\mathbf{b}\mathbf{H} + 2\Omega_6^*(\mathbf{b}\mathbf{H}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{b}\mathbf{H}), \tag{39}$$

and the magnetic induction is given by

$$\mathbf{B} = -2(\Omega_4^*\mathbf{b}\mathbf{H} + \Omega_5^*\mathbf{b}^2\mathbf{H} + \Omega_6^*\mathbf{b}^3\mathbf{H}), \tag{40}$$

where $\Omega^* = \Omega^*(I_1, I_2, K_4, K_5, K_6)$ and Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1,2$, and $\partial\Omega^*/\partial K_i$ for $i = 4,5,6$.

We remark here that if the material properties depend on the *sense* of the magnetic (or magnetic induction) field then an additional invariant would be needed since, for example, Ω would not be invariant under the transformation $\mathbf{B}_l \to -\mathbf{B}_l$. For simplicity we do not consider this possibility here.

4. Rotating tube

4.1. Kinematics

We now apply the equations discussed in Sections 2 and 3 to an incompressible isotropic magnetoelastic circular cylindrical tube spinning about its axis with a constant angular speed ω in the presence of an azimuthal magnetic field. Again we assume, for simplicity, that there are no mechanical body forces. Let the tube in the stress-free reference configuration have internal and external radii A and B, respectively. The geometry is defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z < L, \tag{41}$$

where R, Θ, Z are cylindrical polar coordinates and L is a constant (the length of the tube).

Let the deformed geometry of the tube be given by the equations

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z < l, \tag{42}$$

where r, θ, z are cylindrical polar coordinates in the deformed configuration, l is a constant and circular symmetry is maintained.

Since the material is incompressible the deformation is given by

$$r = \sqrt{\lambda_z^{-1}\left(R^2 - A^2\right) + a^2}, \quad \theta = \Theta + \omega t, \quad z = \lambda_z Z, \tag{43}$$

where λ_z is the (uniform) axial stretch and t denotes time.

The deformation is locally a pure homogeneous strain, the deformation gradient is diagonal with respect to the cylindrical axes, and the principal stretches have the form

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} \equiv \lambda, \quad \lambda_3 = \lambda_z. \tag{44}$$

The corresponding matrix of components of $\mathbf{b}$ is also diagonal, with entries $(\lambda^{-2}\lambda_z^{-2}, \lambda^2, \lambda_z^2)$.

The invariants I_1 and I_2 given by Eq. (34) are computed as

$$I_1 = \lambda^{-2}\lambda_z^{-2} + \lambda^2 + \lambda_z^2, \quad I_2 = \lambda^2\lambda_z^2 + \lambda^{-2} + \lambda_z^{-2}. \tag{45}$$

4.2. Magnetic field

We now assume that the magnetic field is azimuthal, with $\mathbf{B}$ and $\mathbf{H}$ having azimuthal components B_θ and H_θ, respectively, which, because of the assumed symmetry, depend only on r. Since the cylindrical polar axes are principal axes of the deformation, the Lagrangian fields $\mathbf{B}_l$ and $\mathbf{H}_l$ are also azimuthal. Their components are denoted $B_{l\Theta}$ and $H_{l\Theta}$ and the connections $B_\theta = \lambda B_{l\Theta}$ and $H_\theta = \lambda^{-1} H_{l\Theta}$ follow from the specialization of (5).

The invariants I_4, I_5, I_6 and K_4, K_5, K_6 are computed as

$$I_4 = B_{l\Theta}^2, \quad I_5 = \lambda^2 I_4, \quad I_6 = \lambda^4 I_4, \tag{46}$$

and

$$K_4 = H_{l\Theta}^2, \quad K_5 = \lambda^2 K_4, \quad K_6 = \lambda^4 K_4. \tag{47}$$

4.3. Constitutive relations

The only non-zero components of $\boldsymbol{\tau}$ are, in cylindrical polar form, τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz}, and these depend only on r. From Eq. (36) these are calculated as

$$\begin{aligned} \tau_{rr} &= -p + 2\lambda_1^2\left[\Omega_1 + \Omega_2(\lambda^2 + \lambda_z^2)\right], \\ \tau_{\theta\theta} &= -p + 2\lambda^2\left[\Omega_1 + \Omega_2(\lambda_1^2 + \lambda_z^2) + (\Omega_5 + 2\Omega_6\lambda^2)I_4\right], \\ \tau_{zz} &= -p + 2\lambda_z^2\left[\Omega_1 + \Omega_2(\lambda_1^2 + \lambda^2)\right], \end{aligned} \tag{48}$$

while from Eq. (37) we obtain

$$H_\theta = 2\left(\Omega_4\lambda^{-2} + \Omega_5 + \Omega_6\lambda^2\right)B_\theta. \tag{49}$$

Similarly, from Eqs. (39) and (40), we obtain

$$\begin{aligned}\tau_{rr} &= -p^* + 2\lambda_1^2\left[\Omega_1^* + \Omega_2^*(\lambda^2 + \lambda_z^2)\right],\\ \tau_{\theta\theta} &= -p^* + 2\lambda^2\left[\Omega_1^* + \Omega_2^*(\lambda_1^2 + \lambda_z^2) + (\Omega_5^* + 2\Omega_6^*\lambda^2)\lambda^2 K_4\right],\\ \tau_{zz} &= -p^* + 2\lambda_z^2\left[\Omega_1^* + \Omega_2^*(\lambda_1^2 + \lambda^2)\right],\end{aligned} \tag{50}$$

and

$$B_\theta = -2\left(\Omega_4^*\lambda^2 + \Omega_5^*\lambda^4 + \Omega_6^*\lambda^6\right)H_\theta. \tag{51}$$

Since the deformation depends only on λ and λ_z and we have the connections (46) it is convenient to introduce a reduced form of Ω, denoted w, depending only on λ, λ_z and I_4. This is defined by

$$w(\lambda, \lambda_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{52}$$

taken in conjunction with (45) and (46).

The stress differences are now given by the simple expressions

$$\tau_{\theta\theta} - \tau_{rr} = \lambda w_\lambda, \quad \tau_{zz} - \tau_{rr} = \lambda_z w_{\lambda_z}, \tag{53}$$

where the subscripts λ and λ_z signify partial derivatives. Similarly, simplified expressions for the magnetic field and magnetization are obtained as

$$H_\theta = 2\lambda^{-2} w_4 B_\theta, \quad M_\theta = \mu_0^{-1} B_\theta - H_\theta, \tag{54}$$

where $w_4 = \partial w / \partial I_4$.

By defining, similarly, a reduced form w^* of Ω^* by

$$w^*(\lambda, \lambda_z, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{55}$$

with (45) and (47), we obtain from (50) and (51)

$$\tau_{\theta\theta} - \tau_{rr} = \lambda w_\lambda^*, \quad \tau_{zz} - \tau_{rr} = \lambda_z w_{\lambda_z}^*, \tag{56}$$

and

$$B_\theta = -2\lambda^2 w_4^* H_\theta, \quad M_\theta = \mu_0^{-1} B_\theta - H_\theta, \tag{57}$$

where $w_4^* = \partial w^* / \partial K_4$.

4.4. Governing equations and analysis

Since $B_r = 0$ the Eq. $(4)_1$ is satisfied identically and yields no information about B_θ, while equation $(4)_2$ gives

$$H_\theta = \frac{c}{r}, \tag{58}$$

where c is a constant.

The equation of motion (11) reduces to the radial component equation

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho a_r, \tag{59}$$

where $a_r = -\omega^2 r$, which, on use of $(56)_1$, becomes

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \frac{\lambda w_\lambda^*}{r} - \rho\omega^2 r. \tag{60}$$

Here, because H_θ is continuous, it is convenient to use $H_{l\Theta}$ as the independent variable and to work in terms of w^*.

From Eq. (20) the Maxwell stress exterior to the tube (assumed to be vacuum) has components

$$\tau_{mrr} = \tau_{mzz} = -\tau_{m\theta\theta} = -\frac{1}{2}\mu_0 H_\theta^2. \tag{61}$$

Let τ_{mrr} be denoted $\tau_m(r)$ to emphasize its dependence on r. Then, on integrating (60), assuming that there are no mechanical tractions applied on the boundaries $r = a$ and $r = b$, and applying the boundary conditions on these boundaries, we obtain

$$\frac{1}{2}(b^2 - a^2)\rho\omega^2 + \tau_m(b) - \tau_m(a) = \int_a^b \lambda w^*_\lambda \frac{\mathrm{d}r}{r}. \tag{62}$$

Next, on use of (61) and (58), we obtain

$$\frac{1}{2}(b^2 - a^2)\left(\rho\omega^2 + \mu_0 c^2 \frac{b^2 - a^2}{a^2 b^2}\right) = \int_{\lambda_b}^{\lambda_a} \frac{w^*_\lambda}{\lambda^2\lambda_z - 1}\mathrm{d}\lambda, \tag{63}$$

in which we have changed the integration variable from r to λ and introduced the notations $\lambda_a = a/A$, $\lambda_b = b/B$. The latter are connected through

$$(\lambda_b^2\lambda_z - 1)B^2 = (\lambda^2\lambda_z - 1)R^2 = (\lambda_a^2\lambda_z - 1)A^2, \tag{64}$$

which is obtained from the deformation $(43)_1$.

Since H_θ necessarily has the form (58), B_θ is determined by (57). Note that if we had started with B_θ as the 'independent' variable then it would be pre-determined from (54) as a function of r (implicitly and not necessarily uniquely in general), dependent on the form of w, and in this sense it cannot, for this problem, be regarded as an *independent* variable. We can think of H_θ as being generated by an axial steady current, I say, within the core of the tube, in which case $c = I/2\pi$, the value of which is at our disposal. We note in passing that if the azimuthal magnetic field is replaced by an axial field H_z then this is necessarily constant (from equation $(4)_2$) and the Maxwell stress has the same value on each boundary, i.e. $\tau_m(a) = \tau_m(b) = -\mu_0 H_z^2/2$. Eqs. (62) and (63) simplify accordingly since the formulas (53) and (56) hold in this situation also.

The corresponding problem of rotation of a thick-walled tube (with no magnetic effects) has been examined by Haughton and Ogden (1980a), and the specialization of formula (63) in this case was given therein. In the analogous problem of inflation of a thick-walled tube under an internal pressure $P(>0)$ (see, for example, Haughton and Ogden (1979) and Ogden (1997)) with no magnetic field and no rotation, the left-hand side of Eq. (63) is replaced by P and $w^* \equiv w$ depends only on λ and λ_z. The rotation of a tube was also considered by Chadwick et al. (1977), who examined in detail aspects of existence and uniqueness of solution for a freely rotating tube (i.e. a tube rotating in the absence of an applied axial load).

For an elastic material in the absence of a magnetic field the terms in Ω_5^* and Ω_6^* vanish. Standard inequalities such as the Baker-Ericksen inequalities require that $\Omega_1^* + \lambda_z^2\Omega_2^* > 0$. Now, by (64), $\lambda^2\lambda_z - 1$ has the same sign for all r such that $a \leqslant r \leqslant b$. Therefore, if $\lambda^2\lambda_z < 1$ it follows that $w^*_\lambda < 0$ for $a \leqslant r \leqslant b$ and hence the integral is negative (since then $\lambda_b > \lambda_a$). For the integral to be positive we must therefore have $\lambda^2\lambda_z > 1$, and hence $w^*_\lambda > 0$, for $a \leqslant r \leqslant b$ (in which case $\lambda_a > \lambda_b$). Likewise, under pure rotation, again in the absence of magnetic effects, we must have $\lambda^2\lambda_z > 1$ for $a \leqslant r \leqslant b$. One might expect intuitively that the inner radius of the tube increases with the rate of rotation ω and the length of the tube decreases, but the opposite effect is not, in principle, excluded by the inequality $\lambda^2\lambda_z > 1$.

It is interesting to consider, for the purely elastic situation, the strain-energy function given by

$$w^* = w = \frac{\mu}{\alpha}(\lambda^\alpha + \lambda_z^\alpha + \lambda^{-\alpha}\lambda_z^{-\alpha} - 3), \tag{65}$$

where μ and α are material constants satisfying $\mu\alpha > 0$ (see, for example, Ogden (1972, 1997)). We then have

$$\lambda w_\lambda = \mu(\lambda^\alpha - \lambda^{-\alpha}\lambda_z^{-\alpha}), \tag{66}$$

from which it follows that $w_\lambda > 0$ if and only if $\lambda^2\lambda_z > 1$. We shall return to consideration of (65) in connection with the expression for the axial load on the tube.

Turning now to the effect of the magnetic field we note that since both terms on the left-hand side of (63) are positive, the effect of the rotation and the effect of the azimuthal magnetic field are each similar to the effect of internal pressure. In particular, the integral must be positive. However, in the presence of the magnetic field the requirement $w_\lambda^* > 0$ for all $r \in [a, b]$ is no longer necessary. Therefore, in principle, this allows w_λ^* to be negative for some values of r. However, since the constitutive law is different in the present situation, the interpretation can be different compared with the pure pressure case. This can be seen by noting the formula

$$\lambda w_\lambda^* = 2(\lambda^2 - \lambda^{-2}\lambda_z^{-2})(\Omega_1^* + \lambda_z^2\Omega_2^*) + 2\lambda^2 I_4(\Omega_5^* + 2\lambda^2\Omega_6^*), \tag{67}$$

which follows from (48) and (53). Indeed, it is even possible that $\lambda^2\lambda_z < 1$ for $r \in [a, b]$ provided the terms in Ω_5^* and Ω_6^* provide positive contributions to w_λ^* large enough to ensure that the integral is positive. In particular, depending on the form of Ω^*, an azimuthal magnetic field can enhance or counteract the effect of rotation. On its own (without rotation) the magnetic field generates a magnetostrictive effect, which could allow inflation or deflation of the tube coupled with either shortening or lengthening. These various possibilities remain to be tested experimentally for the materials of interest.

We emphasize that for the considered azimuthal magnetic field the analysis applies for a tube of finite length. There is therefore a normal stress on the ends of the tube arising from the component τ_{mzz} of the Maxwell stress given by (61). The traction that must be applied on the ends comes from the difference between the normal stress τ_{zz} within the material and the Maxwell stress τ_{mzz}. The former is given by $(53)_2$.

The resultant axial load, N say, on a cross-section (independent of z), calculated from within the material, is given by

$$N = 2\pi \int_a^b \tau_{zz} r \, dr. \tag{68}$$

After some rearrangement, using (53), (59), then (61) and (58), and noting that the contributions of the Maxwell stresses on $r = a$ and b following integration cancel, we obtain

$$N = \pi \int_a^b (2\lambda_z w_{\lambda_z}^* - \lambda w_\lambda^*) r \, dr + \frac{1}{4}\pi\rho\omega^2(b^4 - a^4). \tag{69}$$

A similar formula was given by Dorfmann and Ogden (in press) for the situation of a tube inflated by an internal pressure in the presence of an azimuthal magnetic field. The corresponding resultant, N_m say, of τ_{mzz} on the ends of the tube is calculated as

$$N_m = -\pi\mu_0 c^2 \log(b/a). \tag{70}$$

From (67) we see that if there is no magnetic field and the length of the tube is held fixed so that $\lambda_z = 1$ then necessarily $\lambda > 1$ under rotation. After a short calculation it can then be seen that the sign of the integrand in (69) is that of $\Omega_2^* - \Omega_1^*$, which, depending on the choice of material model, can be positive or negative. Thus, while the term in ω gives a positive contribution to N the value of N can be either positive or negative. This suggests that either contraction or extension of the tube may be associated with the rotation, as also concluded by Chadwick et al. (1977).

It is interesting to examine the sign of the integrand in (69) for the strain-energy function (65) in the purely elastic case. We calculate

$$2\lambda_z w_{\lambda_z} - \lambda w_\lambda = \mu\lambda^{-\alpha}\lambda_z^{-\alpha}(2\lambda^\alpha\lambda_z^{2\alpha} - \lambda^{2\alpha}\lambda_z^\alpha - 1). \tag{71}$$

To illustrate the result we plot, in Fig. 1, the curve of $2\lambda^\alpha\lambda_z^{2\alpha} - \lambda^{2\alpha}\lambda_z^\alpha = 1$ in (λ_z, λ) space for $\alpha = 2$ (short-dashed curve) compared with $\lambda^2\lambda_z = 1$ (continuous curve). We also plot the corresponding curve for $\alpha = -2$ (long-dashed curve), which may be expressed as $2\lambda^2 - \lambda_z^2 - \lambda^4\lambda_z^4 = 0$. For $\mu > 0$ the sign of (71) is positive to the right of the short-dashed curve. For any other $\alpha > 0$ such that $\mu\alpha > 0$ the curve is qualitatively similar and again (71) is positive to the right. Likewise, for $\alpha = -2$ (and $\mu < 0$), (71) is positive to the right of the long-dashed curve and similarly for other negative values of α. In each case (71) is negative to the left of the relevant dashed curve. Note that for $\lambda_z = 1$ the expression in (71) reduces to $-\mu(\lambda^\alpha - \lambda^{-\alpha})^2$. Thus, on this line (71) is negative for $\mu > 0$ and positive for $\mu < 0$. This reinforces the discussion in the foregoing paragraph concerning the case $\lambda_z = 1$. In particular, if $\mu < 0$ (and $\alpha < 0$) then N is necessarily positive, so that an axial tension is required to prevent the tube shortening by rotation. On the other hand, if $\mu > 0$, as for the neo-Hookean material ($\alpha = 2$), axial compression ($N < 0$) is possible, which is not inconsistent with a tendency for the rotation to lengthen the tube.

If λ_z is fixed at a value greater than 1 then, clearly, by reference to Fig. 1, it can be seen that the integrand (71) is positive for all the considered strain-energy functions with $\alpha < 0$, and also for those with $\alpha > 0$ up to values of λ where the short-dashed curve is crossed. Then, N is necessarily tensile. The situation is different for values of $\lambda_z < 1$. For $\alpha > 0$ the integrand (71) is then negative, while for $\alpha < 0$ it is negative until, as λ increases, the long-dashed curve is reached.

If we now include the effect of the magnetic field but set $\omega = 0$ then, on taking account of the exterior Maxwell stress, we may write the integrand in the resultant $N - N_m$ (for $\lambda_z = 1$, for example) as

$$2(\lambda^2 + \lambda^{-2} - 2)(\Omega_2^* - \Omega_1^*) - 2\lambda^2 K_4(\Omega_5^* + 2\lambda^2\Omega_6^*) + \mu_0\lambda^{-2}K_4. \tag{72}$$

Thus, if

$$2\lambda^4(\Omega_5^* + 2\lambda^2\Omega_6^*) > \mu_0, \tag{73}$$

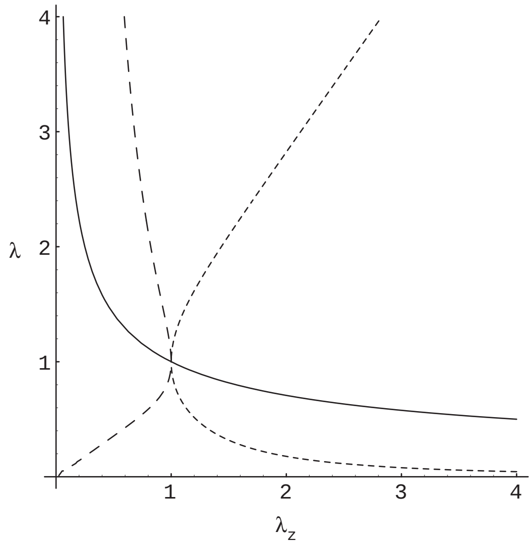

Fig. 1. Plot of $\lambda^2\lambda_z = 1$ (continuous curve), $2\lambda^2\lambda_z^4 - \lambda^4\lambda_z^2 = 1$ (short-dashed curve, associated with $\alpha = 2$), and $2\lambda^2 - \lambda_z^2 - \lambda^4\lambda_z^4 = 0$ (long-dashed curve, associated with $\alpha = -2$) in (λ_z, λ) space. The expression (71) is positive (negative) to the right (left) of the relevant dashed curve in each case.

the magnetic field generates a negative contribution to $N - N_m$, suggesting the tendency of the magnetic field to lengthen the tube. This makes physical sense, as the following argument, adapted from that in Kankanala and Triantafyllidis (2004) relating to an axial magnetic field in a cylinder, indicates. As a result of an azimuthal magnetic field, magnetic particles at a given radius will tend to be attracted circumferentially, thereby reducing the radius and tending to increase the length of the tube.

5. Rotating solid cylinder

Consider now a solid cylinder with circular cross-section rotating with constant angular speed ω about its axis. We again use cylindrical polar coordinates in the reference and current configurations and define the cross-section of the cylinder in the reference configuration by the inequalities

$$0 \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi. \tag{74}$$

The cylinder is deformed by the combination of its rotation and an applied axial magnetic field in such a way that the cross-section remains circular and uniform. The deformed cross-section is given by

$$0 \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{75}$$

In component form the (homogeneous) deformation is given by

$$r = \lambda_z^{-1/2} R, \quad \theta = \Theta + \omega t, \quad z = \lambda_z Z, \tag{76}$$

with the same notation as was used in (43). With respect to the cylindrical polar axes the deformation gradient $\mathbf{F}$ is diagonal with entries $(\lambda_z^{-1/2}, \lambda_z^{-1/2}, \lambda_z)$, while the left Cauchy-Green deformation tensor is coaxial with the cylindrical polar axes and has principal values $(\lambda_z^{-1}, \lambda_z^{-1}, \lambda_z^2)$. The principal invariants I_1 and I_2 are simply

$$I_1 = \lambda_z^2 + 2\lambda_z^{-1}, \quad I_2 = 2\lambda_z + \lambda_z^{-2}. \tag{77}$$

Let the (axial) components of $\mathbf{B}$ and $\mathbf{H}$ be denoted by B_z and H_z, respectively. Their Lagrangian counterparts, denoted B_{lZ} and H_{lZ}, are then given by the appropriate specialization of (5) through

$$B_z = \lambda_z B_{lZ}, \quad H_z = \lambda_z^{-1} H_{lZ}, \tag{78}$$

and the invariants I_4, I_5, I_6 and K_4, K_5, K_6 are given by

$$I_4 = B_{lZ}^2, \quad I_5 = \lambda_z^2 I_4, \quad I_6 = \lambda_z^4 I_4, \tag{79}$$

and

$$K_4 = H_{lZ}^2, \quad K_5 = \lambda_z^2 K_4, \quad K_6 = \lambda_z^4 K_4. \tag{80}$$

The components of τ are calculated in a similar way to those in Eqs. (48) and (50) and we do not therefore give their counterparts here. Similarly for the equations for H_z and B_z. In the present problem there is just one independent deformation variable, namely λ_z. In view of the connections (79) and (80) we again introduce reduced forms of Ω and Ω^*, denoted by w and w^* respectively, but now defined by

$$w(\lambda_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{81}$$

with (77) and (79), and

$$w^*(\lambda_z, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{82}$$

with (77) and (80).

It follows that $\tau_{\theta\theta} = \tau_{rr}$,

$$\tau_{zz} - \tau_{rr} = \lambda_z w_{\lambda_z} = \lambda_z w^*_{\lambda_z}, \tag{83}$$

and

$$H_z = 2\lambda_z^{-2} w_4 B_z, \quad B_z = -2\lambda_z^2 w_4^* H_z, \tag{84}$$

while M_z is given by $M_z = \mu_0^{-1} B_z - H_z$.

Since $\tau_{\theta\theta} = \tau_{rr}$, the equilibrium equation (59) reduces to

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \rho\omega^2 r = 0. \tag{85}$$

Next, recalling that the tangential component H_z of the magnetic field must be continuous across the boundary $r = b$, we calculate the Maxwell stress $\boldsymbol{\tau}_m$ exterior to the cylinder from (20). Its (uniform) components are

$$\tau_{mrr} = \tau_{m\theta\theta} = -\tau_{mzz} = -\frac{1}{2}\mu_0 H_z^2. \tag{86}$$

On integration of (85) and use of (86) and continuity of τ_{rr} across the boundary $r = b$, we obtain

$$\tau_{rr} = \frac{1}{2}\rho\omega^2(b^2 - r^2) - \frac{1}{2}\mu_0 H_z^2. \tag{87}$$

The axial stress is then obtained from Eq. (83) in the form

$$\tau_{zz} = \lambda_z w_{\lambda_z}^* + \frac{1}{2}\rho\omega^2(b^2 - r^2) - \frac{1}{2}\mu_0 H_z^2, \tag{88}$$

and its resultant on a cross-section of the cylinder, denoted N, is given by

$$N/\pi b^2 = \lambda_z w_{\lambda_z}^* + \frac{1}{4}\rho\omega^2 b^4 - \frac{1}{2}\mu_0 H_z^2. \tag{89}$$

The consequences of (89) for a rotating elastic cylinder (with no magnetic field) for $N = 0$ were examined by Chadwick et al. (1977). It was shown, in particular, that rotation is necessarily accompanied by shortening of the cylinder provided the reasonable assumption that $w_{\lambda_z} \geqslant 0 (\leqslant 0)$ corresponds to $\lambda_z \geqslant 1 (\leqslant 1)$ is adopted, as is usually required for uniaxial stress. Haughton and Ogden (1980b) considered the elastic specialization of (89) for different values of N in respect of a particular class of strain-energy functions. They showed that for fixed values of $N < 0$ the curves of ω^2 against λ_z were monotonic, while for $N \geqslant 0$ they were non-monotonic. This also applies in the presence of a magnetic field under appropriate conditions.

If, as in Section 4, we consider $\lambda_z = 1$ then (88) becomes

$$\tau_{zz} = 2(\Omega_5^* + 2\Omega_6^*)K_4 - \frac{1}{2}\mu_0 K_4 + \frac{1}{2}\rho\omega^2(B^2 - R^2). \tag{90}$$

In this case, when there is no magnetic field, the rotation generates a tensile axial load irrespective of the form of constitutive law, in contrast to the result in Section 4.4. In the absence of rotation, the magnetic field (which is in this case axial) also generates a tensile axial load if the analogue $2(\Omega_5^* + 2\Omega_6^*) > \mu_0$ of the inequality (73) holds. Thus, the axial field has the opposite effect on the cylinder (and similarly on a tube) to that of an azimuthal field on the tube in terms of mechanical response.

Formally, the analysis in this section applies to an infinitely long cylinder since a *uniform* axial magnetic field cannot be maintained in a cylinder of finite length. This is because H_z is continuous across $r = b$ whereas B_z is continuous across the ends of the cylinder. These two requirements are not in general consistent with the (vacuum) relation $B_z = \mu_0 H_z$. Thus, for a cylinder of finite length the magnetic field is necessarily inhomogeneous. However, for sufficiently large aspect ratio length/radius it can be taken as approximately uniform except near the ends, and this allows the possibility of comparison between the theory and experimental results.

6. Concluding remarks

A spinning elastomeric cylinder can be regarded as a simple model of many devices of technical interest. For example, when the rotating tube is mounted on a rigid spindle (of circular cylindrical geometry) the internal radius a in the deformed configuration coincides with the radius of the spindle. Of particular interest is the contact stress between the tube and spindle, which must satisfy the shrink-fit inequality $\tau_{rr}(a) < 0$. If this inequality is not satisfied then dangerous effects (which may be enhanced by resonance phenomena within a dynamical framework) may be initiated. Understanding of the influence of a magnetic field may then allow control of $\tau_{rr}(a)$ so as to ensure that the shrink-fit condition is satisfied.

It is clear from (61) and the continuity conditions that the contribution of the magnetic field to τ_{rr} due to the Maxwell stress in the spindle (assumed non-magnetic) is negative, and it is therefore possible in principle to tune H_θ so as to ensure that the shrink-fit inequality is satisfied in situations where there is a mechanical spin-off speed. In a purely mechanical theory (see Chadwick et al. (1977) for a detailed discussion) it is known that for some particular constitutive equations there exists a critical value of the speed of rotation for which $\tau_{rr}(a) = 0$ and the tube loses contact with the spindle. This is the *spin-off* speed.

In the present situation, as already noted, we may think of H_θ as being generated by an axial steady current I, which determines the constant c in (58), and we may therefore effectively control the magnetic field. Moreover, to have a clear picture of the behaviour we need to consider particular forms of the constitutive equations in order to evaluate the right-hand side of (63). But, as we have already discussed, the effect of the magnetic field is similar to that of an internal pressure and it is this that allows the application of a suitable magnetic field to bring the spin-off speed to an acceptable value.

The theory considered in this paper is phenomenological and these possibilities therefore have to be tested experimentally since we have no a priori information about the constitutive parameters involved. Without experimental investigations it is not possible to appreciate if the *positive* influence of a magnetic field is obtained for a range of speeds of technical interest and with the application of technically feasible currents. The results presented here in the framework of a nonlinear theory of magnetoelasticity are the kind of rigorous computations that are needed for the rational guidance of experimental tests. Indeed, the findings here may provide a basis for the use of *active* materials in improving existing devices and transforming them into *smart structures*.

In Patterson and Hill (1977) the stability of a rotating solid cylinder of neo-Hookean elastic material was examined, while Haughton and Ogden (1980a,b) discussed the stability and bifurcation of rotating tubes and cylinders of isotropic elastic material in some detail. It is also of interest to examine the influence of a magnetic field on such behaviour, and this will be the subject of a separate communication.

References

Chadwick, P., Creasy, C.F.M., Hart, V.G., 1977. The deformation of rubber cylinders and tubes by rotation. Journal of the Australian Mathematical Society B 20, 62–96.

Dorfmann, A., Ogden, R.W., 2003a. Magnetoelastic modelling of elastomers. European Journal of Mechanics A/Solids 22, 497–507.

Dorfmann, A., Ogden, R.W., 2003b. Nonlinear magnetoelastic deformations of elastomers. Acta Mechanica 167, 13–28.

Dorfmann, A., Ogden, R.W., 2004. Nonlinear magnetoelastic deformations. Quarterly Journal of Mechanics and Applied Mathematics 57, 599–622.

Dorfmann, A., Ogden, R.W., in press. Some problems in nonlinear magnetoelasticity. Zeitschrift für Angewandte Mathematik und Physik.

Haughton, D.M., Ogden, R.W., 1979. Bifurcation of inflated circular cylinders of elastic material under axial loading II. Exact theory for thick-walled tubes. Journal of the Mechanics and Physics of Solids 27, 489–512.

Haughton, D.M., Ogden, R.W., 1980a. Bifurcation of rotating thick-walled elastic tubes. Journal of the Mechanics and Physics of Solids 28, 59–74.

Haughton, D.M., Ogden, R.W., 1980b. Bifurcation of finitely deformed rotating elastic cylinders. Quarterly Journal of Mechanics and Applied Mathematics 33, 251–265.

Holzapfel, G.A., 2001. Nonlinear Solid Mechanics: a Continuum Approach for Engineering, second ed. John Wiley, Chichester.

Hutter, K., van de Ven, A.A.F., 1978Field Matter Interactions in Thermoelastic Solids, Lecture Notes in Physics, Vol. 88. Springer, Berlin.

Kankanala, S.V., Triantafyllidis, N., 2004. On finitely strained magnetorheological elastomers. Journal of the Mechanics and Physics of Solids 52, 2869–2908.

Kovetz, A., 2000. Electromagnetic Theory. Oxford University Press, Oxford.

Maugin, G.A., 1988. Continuum Mechanics of Electromagnetic Solids. North Holland, Amsterdam.

Ogden, R.W., 1972. Large deformation isotropic elasticity I: on the correlation of theory and experiment for incompressible rubberlike solids. Proceedings of the Royal Society of London A 326, 565–584.

Ogden, R.W., 1997. Non-linear Elastic Deformations. Dover, New York.

Ogden, R.W., 2001. Elements of the theory of finite elasticity. In: Fu, Y.B., Ogden, R.W. (Eds.), Nonlinear Elasticity: Theory and Applications, London Mathematical Society Lecture Notes, Vol. 283. Cambridge University Press, Cambridge, pp. 1–57.

Patterson, J.C., Hill, J.M., 1977. The stability of a solid rotating neo-Hookean cylinder. Mechanics Research Communications 4, 69–74.

Spencer, A.J.M., 1971. Theory of Invariants. In: Eringen, A.C. (Ed.), Continuum Physics, Vol. 1. Academic Press, New York, pp. 239–353.

Steigmann, D.J., 2004. Equilibrium theory for magnetic elastomers and magnetoelastic membranes. International Journal of Nonlinear Mechanics 39, 1193–1216.